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ON THE CONSTRUCTION OF MINIMAL SKEW PRODUCTS

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ABSTRACT

It is shown that under fairly general conditions on a compact metric space Y there are minimal homeomorphisms on $Z \times Y$ of the form $(z, y) \to (\sigma z, h_z(y))$ where (Z, σ) is an arbitrary metric minimal flow and $z \to h_z$ is a continuous map from Z to the space of homeomorphisms of Y. Similar results are obtained for strict ergodicity, topological weak mixing and some relativized concepts.

1. Introduction and statement of results

Using an idea due to Anosov and exploited by A. B. Katok [1], and by A. Fathi and M. Herman [4], we show that under fairly general conditions on a compact metric space Y, there are minimal homeomorphisms on $X = Z \times Y$, of the form $(z, y) \to (\sigma z, h_z(y))$, i.e. skew products with σ. Here (Z, σ) is an arbitrary compact metric minimal flow and $z \to h_z$ is a continuous map from Z to the space of homeomorphisms of Y.

In particular as we shall see, the Hilbert cube satisfies the condition required of Y, and thus we answer a question listed in [2]. Similar results about unique ergodicity, topological weak mixing, relative topological weak mixing and relative proximality are established.

For related results see [3], [9], [5], [7], [8], [6]. We repeat now the basic definitions and establish the notation that will be in force.

Let $\mathcal{G}$ be a topological group and let Y be a compact space. If $\mathcal{G}$ acts on Y as a group of homeomorphisms and the action is jointly continuous we call the couple $(Y, \mathcal{G})$ a *flow*. If $\mathcal{G}$ is generated by a single element T we write (Y, T) instead of $(Y, \mathcal{G})$. A flow $(Y, \mathcal{G})$ is called *topologically ergodic* if given two non-empty open subsets U and V of Y, there exists $g \in \mathcal{G}$ such that $gU \cap V \neq \emptyset$. $(Y, \mathcal{G})$ is *topologically weakly mixing* if $(Y \times Y, \mathcal{G})$ is topologically

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ergodic. $(Y, \mathscr{G})$ is *minimal* if $\mathscr{G}y$ is dense in Y for each $y \in Y$, equivalently $(Y, \mathscr{G})$ is minimal iff $\{gU: g \in \mathscr{G}\}$ is a covering of Y, whenever U is a non-empty open subset of Y.

A flow $(Y, \mathscr{G})$ is *uniquely ergodic* if there is one and only one $\mathscr{G}$-invariant Borel probability measure on Y. It is called *strictly ergodic* if in addition this unique invariant measure is positive on every non-empty open subset of Y. Since the support of an invariant measure is a closed invariant subset, strict ergodicity follows from unique ergodicity plus minimality. When $\mathscr{G}$ is generated by a single homeomorphism T, then (Y, T) is uniquely ergodic iff $(1/(n+1))\sum_{k=0}^{n} f(T^k x)$ converges uniformly to a constant, for every continuous function f on Y. A pair of points $y_1, y_2 \in Y$ is called *proximal* if for every neighbourhood V of the diagonal in $Y \times Y$ there exists $g \in \mathscr{G}$ such that $g(y_1, y_2) \in V$.

Let $(Z, \mathscr{G})$ be another flow and let $\phi: Z \to Y$ be a continuous onto map such that for every $z \in Z$ and $g \in \mathscr{G}$ $\phi(gz) = g\phi(z)$. Then $(Z, \mathscr{G})$ is an *extension* of $(Y, \mathscr{G})$. Let

$$L = \{(z_1, z_2) \in X \times Z: \phi(z_1) = \phi(z_2)\};$$

the extension is called *topologically weak mixing* if $(L, \mathscr{G})$ is topologically ergodic. The extension is called *proximal* if $(z_1, z_2) \in L$ implies z_1 and z_2 are proximal.

When the group $\mathscr{G}$ of the flow is the group of reals (denoted by $\mathbf{R}$) we say that the flow is *real*. $\mathbf{T}$ denotes the one torus (considered as the group of reals modulo the integers). $I = [0, 1]$ is the unit interval and λ denotes Lebesgue measure on I. The letter d will denote a metric on each of the metric spaces under consideration. $\mathbf{N}$ denotes the set of natural numbers. Q is the Hilbert cube i.e. $Q = I^{\mathbf{N}}$ equipped with the metric $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|/2^i$.

Whenever M is a compact metric space $\mathscr{H}(M)$ will denote the space of all homeomorphisms of M equipped with the metric

$$d(g, h) = \operatorname*{Sup}_{m \in M} d(g(m), h(m)) + \operatorname*{Sup}_{m \in M} d(g^{-1}(m), h^{-1}(m)).$$

With this metric $\mathscr{H}(M)$ is a complete metric space and a topological group.

Let (Z, σ) be a compact metric minimal infinite flow, let Y be a compact metric space and let $X = Z \times Y$. We consider the subspace $\mathscr{O}(\sigma) \subset \mathscr{H}(X)$ where

$$\mathscr{O}(\sigma) = \{G^{-1} \circ \sigma \circ G: G \in \mathscr{H}(X)\}.$$

(Here σ is identified with $\sigma \times \mathrm{id}$, where id is the identity map on Y.) Let $\mathscr{H}_s(X)$ be the subgroup of $\mathscr{H}(X)$ which consists of homeomorphisms which fixes all subspaces of X of the form $\{z\}\times Y$ $(z \in Z)$. Such a homeomorphism G is determined by a continuous map $z \to g_z$ of Z into $\mathcal{H}(Y)$, by $G(z,y) = (z, g_z(y))$. Put

$$\mathcal{S}(\sigma) = \{G^{-1}\circ\sigma\circ G \colon G \in \mathcal{H}_s(X)\}.$$

If $\mathcal{G}$ is a subgroup of $\mathcal{H}(Y)$, let $\mathcal{G}_s \subset \mathcal{H}_s(X)$ be the subgroup of those elements of $\mathcal{H}_s(X)$ which come from continuous maps $z \to g_z$ of Z into $\mathcal{G}$. Put

$$\mathcal{S}_{\mathcal{G}}(\sigma) = \{G^{-1}\circ\sigma\circ G \colon G \in \mathcal{G}_s\}.$$

We now state our results which will be proven in the subsequent sections. Some remarks and examples follow the statement of each theorem.

THEOREM 1. *Let $\mathcal{G}$ be a subgroup of $\mathcal{H}(Y)$ which is pathwise connected and such that $(Y,\mathcal{G})$ is a minimal flow. Then for a residual subset $\mathcal{R} \subset \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$, (X,T) is a minimal flow for every $T \in \mathcal{R}$. A similar statement holds for $\overline{\mathcal{O}(\sigma)}$ instead of $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$.*

We notice that the elements of $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$ are limits of transformations of X of the form

$$(z,y) \to (\sigma z, g_{\sigma z}^{-1} g_z(y)),$$

where $\{g_z\}_{z\in Z} \subset \mathcal{G}$. Thus an element $T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$ has the form $(z,y) \xrightarrow{T} (\sigma z, h_z(y))$ where $z \to h_z$ is a continuous map of Z into $\bar{\mathcal{G}}$. In particular when $\mathcal{G}$ is a compact abelian group the minimal flows obtained will be group extensions of (Z,σ).

Let (Z,σ) be an arbitrary minimal compact metric infinite flow and let $Y = M^n$ be a compact connected n-dimensional manifold. Then it is clear that the identity path component $\mathcal{G}$ of $\mathcal{H}(M^n)$ acts transitively on M^n and Theorem 1 applies. More generally let Q be the Hilbert cube and let M be any compact connected Q-manifold. By [2, theorem 19.4] the identity path component $\mathcal{G}$ of $\mathcal{H}(M)$ acts transitively on M, and again we can apply Theorem 1 to obtain a minimal flow on $Z\times M$. In particular when $(Z,\sigma) = (\mathbf{T}, R_\alpha)$, an irrational rotation of the circle, we have a minimal homeomorphism on $\mathbf{T}\times M$. This answers one of the open problems listed in [2].

Finally let $(Y,\mathbf{R})$ be any minimal metric real flow. Then clearly the conditions in Theorem 1 are satisfied for $\mathcal{G} = \mathbf{R} \subset \mathcal{H}(Y)$.

THEOREM 2. *Let (Z,σ) be a strictly ergodic infinite metric flow; let $\mathcal{G}$ be a pathwise connected subgroup of $\mathcal{H}(Y)$ with either one of the following properties:*

(A) *There exists a basis $\mathcal{U}$ for open sets in Y such that for every $U \in \mathcal{U}$ and $\varepsilon > 0$*

there are homeomorphisms $h_1, \cdots, h_M$ in $\mathscr{G}$ such that

$$\frac{1}{M}\sum_{j=1}^{M} 1_{h_j(Y\setminus U)}(y) \leqq \varepsilon,$$

for all $y \in Y$. (1_D is the characteristic function of the subset D.)

(B) *$\mathscr{G}$ is a compact group acting transitively on Y and as a topological space it is a continuous image of the unit interval, i.e., a Peano space.*
Then for a residual subset $\mathscr{R} \subset \overline{\mathscr{S}_{\mathscr{G}}(\sigma)}$, (X, T) is strictly ergodic for every $T \in \mathscr{R}$.

Let us consider more closely the condition (A). Suppose that our flow $(Y, \mathscr{G})$ is such that for every $M \in \mathbf{N}$ and every $U \in \mathscr{U}$ there are homeomorphisms $h_1, h_2, \cdots, h_M \in \mathscr{G}$ for which $\{h_j(Y\setminus U)\}_{j=1}^{M}$ is a family of pairwise disjoint subsets. (This is for example the case when we let $Y = \mathbf{P}^1$, the projective line, and $\mathscr{G} = \mathrm{SL}(2, \mathbf{R})$; or when $Y = Q$ and $\mathscr{G}$ is the path component of the identity in $\mathscr{H}(Q)$). Then clearly condition (A) is satisfied. However, when $Y = \mathbf{P}^n$, the projective n-space, $\mathscr{G} = \mathrm{SL}(n+1, \mathbf{R})$ and $n \geqq 2$, this stronger condition is not satisfied while condition (A) still holds.

To see this we observe that the complement of a small neighbourhood of a point in $\mathbf{P}^n$ can be squeezed, by an element of $\mathscr{G}$, to a small neighbourhood of an $n-1$ dimensional hyperplane, and the intersection of any $n+1$ such hyperplanes in a general position is empty.

Choosing $\varepsilon = \frac{1}{2}$ and $U \in \mathscr{U}$ we see that for every $y \in Y$, $y \in \bigcup_{j=1}^{M} h_j(U)$. Thus condition (A) implies the minimality of $(Y, \mathscr{G})$.

THEOREM 3. *Let $\mathscr{G}$ be a pathwise connected subgroup of $\mathscr{H}(Y)$ with the following property: for every pair of points $y_1, y_2 \in Y$ there exist neighbourhoods U and V of y_1 and y_2 respectively, such that for every $\varepsilon > 0$ there exists $h \in \mathscr{G}$ with* $\operatorname{diam}(h(V \cup U)) < \varepsilon$. *Then for a residual subset $\mathscr{R} \subset \overline{\mathscr{S}_{\mathscr{G}}(\sigma)}$, (X, T) is a proximal extension of (Z, σ), for every $T \in \mathscr{R}$.*

THEOREM 4. *Let $\mathscr{G}$ be a pathwise connected subgroup of $\mathscr{H}(Y)$ such that $(Y, \mathscr{G})$ is topologically weak mixing. Then for a residual subset $\mathscr{R} \subset \overline{\mathscr{S}_{\mathscr{G}}(\sigma)}$, (X, T) is a topologically weak mixing extension of (Z, σ) for every $T \in \mathscr{R}$.*

Let $(Y, \mathscr{G})$ be $(\mathbf{P}^n, \mathrm{SL}(n+1, \mathbf{R}))$ or $(Q, \mathscr{G})$ where $\mathscr{G}$ is the identity path component of $\mathscr{H}(Q)$. Then it is easy to check that the condition in Theorem 3 holds. Clearly the action of $\mathrm{SL}(n+1, \mathbf{R})$ on $\mathbf{P}^n$ is doubly transitive and by [2, Theorem 19.4] this is also true for the action of $\mathscr{G}$ on Q. In particular in both cases $(Y, \mathscr{G})$ is topologically weak mixing. Thus for an arbitrary minimal infinite metric flow (Z, σ), there are many minimal homeomorphisms of $Z \times Y$, where

$Y = \mathbf{P}^n$ or Q, which are minimal, strictly ergodic, weakly mixing and proximal extensions of (Z, σ).

In particular if we let $(Z, \sigma) = (\mathbf{T}, R_\alpha)$, an irrational rotation of the circle, and $(Y, \mathcal{G}) = (\mathbf{P}^1, \mathrm{SL}(2, \mathbf{R}))$, then since $\mathbf{P}^1$ is homeomorphic to $\mathbf{T}$, we obtain a minimal flow on the torus $\mathbf{T}^2$, which is a proximal extension of $(\mathbf{T}, R_\alpha)$, and is not an almost one to one extension. This answers a question of H. Furstenberg about the existence of such flows. (See also [8] and [5].)

THEOREM 5. *Let $Z = \mathbf{T}$ and $\sigma = R_\alpha$ an irrational rotation. Let $\mathcal{G}$ be the identity path component of $\mathcal{H}(Y)$ and assume that Y does not reduce to a single point and that $(Y, \mathcal{G})$ is topologically ergodic. Then for a residual subset $\mathcal{R} \subset \overline{\mathcal{O}(\sigma)}$, (X, T) is topologically weak mixing for every $T \in \mathcal{R}$.*

Unlike the previous theorems, the topologically weakly mixing homeomorphisms, whose existence is stated in Theorem 5, can not be skew products of X over $(\mathbf{T}, R_\alpha)$, since a topologically weak mixing flow can not admit a non-trivial equicontinuous factor. It is interesting to notice, however, that these topologically weak mixing homeomorphisms are elements of $\overline{\mathcal{O}(R_\alpha)}$, i.e. they are the uniform limits of conjugations of R_α by homeomorphisms of X.

Since the action of the identity path component $\mathcal{G}$, of $\mathcal{H}(M)$ — where M is a connected compact n-dimensional or a Hilbert cube manifold — is transitive on M, $(M, \mathcal{G})$ is *a fortiori* topologically ergodic and our theorem can be applied to $\mathbf{T} \times M$.

We do not know whether Theorem 5 can be generalized to an arbitrary minimal, infinite, metric flow (Z, σ). The proofs of Theorems 1–5 will now be given. Their structure is very similar, in all cases a translation of the hypotheses and conclusions reduces the proof to the construction of maps satisfying quite specific properties. We have refrained from the temptation to find some master theorem that would contain these results as special cases — the price we pay is a certain unavoidable repetition in the arguments.

2. Minimality

A PROOF OF THEOREM 1. Let U be a non-empty open subset of X. As in [4] denote

$$E_U = \left\{ T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)} \colon \bigcup_{i=0}^{\infty} T^i U = X \right\}.$$

Clearly E_U is an open subset of $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$. If $\{U_i\}$ is a countable basis for open sets in X then $\mathcal{R} = \bigcap E_{U_i}$ consists precisely of the minimal elements of $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$. We now show that E_U is dense in $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$, and then the Baire theorem will complete the proof.

Let $G \in \mathcal{G}_s$, then

$$\begin{aligned} GE_UG^{-1} &= \{G \circ T \circ G^{-1}: T \in E_U\} \\ &= \left\{S \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}: \bigcup_{i=0}^{\infty} S^i GU = X\right\} \\ &= E_{GU}. \end{aligned}$$

Since $\mathcal{H}(X)$ is a topological group we have $G\overline{E_U}G^{-1} = \overline{E_{GU}}$.

In order to prove that E_U is dense it suffices to show that for every $G \in \mathcal{G}_s$, $G^{-1} \circ \sigma \circ G \in \overline{E_U}$, or equivalently that $\sigma \in G\overline{E_U}G^{-1} = \overline{E_{G^{-1}U}}$. Since G and U are arbitrary all we have to show is that $\sigma \in \overline{E_U}$. This will follow from the following lemma.

2.1. Lemma. *Given $\varepsilon > 0$ there exists $G \in \mathcal{G}_s$ such that*

(1) $d(\sigma, G^{-1} \circ \sigma \circ G) < \varepsilon$,
(2) $G^{-1} \circ \sigma \circ G \in E_U$.

Proof. Let $W \subset Z$ and $V \subset Y$ be two non-empty open sets such that $W \times V \subset U$. Since $(Y, \mathcal{G})$ is minimal there is a finite subset $\{h_0, h_{1/n}, \cdots, h_{n-1/n}\} \subset \mathcal{G}$ such that $\bigcup_{i=0}^{n-1} h_{i/n} V = Y$. Let $t \to h_t$ be an extension to a continuous function from I to $\mathcal{G}$. There exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $d(h_{t_1}^{-1} h_{t_2}, id) < \varepsilon$. Let $m \in \mathbf{N}$ satisfy $2/m < \delta$. Next we choose an open subset $A \subset W$ such that $A, \sigma A, \cdots, \sigma^{m-1} A$ are pairwise disjoint (here we use the fact that (Z, σ) is infinite). Let K be a subset of A which is homeomorphic to a Cantor set, and let $\tilde{\theta}: K \to I$ be a continuous onto map. Define $\tilde{\theta}$ on $\bigcup_{i=0}^{m-1} \sigma^i K$ by

$$\tilde{\theta}(z) = \tilde{\theta}(\sigma^{-i} z) \qquad \text{if } z \in \sigma^i K.$$

Now extend $\tilde{\theta}$ to a continuous map $\tilde{\theta}: Z \to I$. Finally we define $\theta: Z \to I$ by

$$\theta(z) = \frac{1}{m} \sum_{i=0}^{m-1} \tilde{\theta}(\sigma^i z).$$

Clearly $\theta | K = \tilde{\theta} | K$ and this implies that θ maps W onto I. Let $g: Z \to \mathcal{G}$ be defined by $g_z = h_{\theta(z)}$ and put $G(z, y) = (z, g_z(y))$. We claim that G satisfies both (1) and (2). For $(z, y) \in X$

$$G^{-1}\circ\sigma\circ G(z,y)=(\sigma z, g_{\sigma z}^{-1}g_z(y))$$
$$=(\sigma z, h_{\theta(\sigma z)}^{-1}h_{\theta(z)}(y)).$$

But

$$|\theta(\sigma z)-\theta(z)|=\frac{1}{m}|\tilde{\theta}(\sigma^m z)-\tilde{\theta}(z)|\leqq\frac{2}{m}<\delta.$$

Hence $d(g_{\sigma z}^{-1}g_z(y),y)<\varepsilon$ and therefore $d(G^{-1}\circ\sigma\circ G,\sigma)<\varepsilon$.

To see (2) it suffices to show that $\bigcup_{i=0}^{\infty}\sigma^i GU=X$. Now the set GU has the following property: for each $y\in Y$ $GU\cap Z\times\{y\}\neq\emptyset$. In fact since $\bigcup_{i=0}^{n-1}h_{i/n}V=Y$ there exists an i such that $y\in h_{i/n}V$ and if we choose $z\in W$ for which $\theta(z)=i/n$ we have

$$G^{-1}(z,y)=(z,h_{i/n}^{-1}(y))\in W\times V\subset U.$$

The minimality of (Z,σ) implies that for every $y\in Y$ there exists k for which

$$Z\times\{y\}\subset GU\cup\sigma GU\cdots\cup\sigma^k GU,$$

hence $\bigcup_{i=0}^{\infty}\sigma^i GU=X$ and the proof of the lemma is complete. The proof for $\overline{\mathcal{O}(\sigma)}$ is analogous and this completes the proof of Theorem 1.

3. Unique ergodicity

A Proof of Theorem 2. For $f\in C(X)$ and $\varepsilon>0$ let

$$E_{f,\varepsilon}=\left\{T\in\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}:\exists n\,\&\,c,\left\|\frac{1}{n+1}\sum_{k=0}^{n}f(T^kx)-c\right\|<\varepsilon\right\}.$$

Clearly $E_{f,\varepsilon}$ is open and it is easy to check that $\mathcal{R}=\bigcap_{i,j}E_{f_i,1/j}$, where $\{f_i\}$ is a countable dense subset of $C(X)$, consists precisely of the uniquely ergodic transformations in $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$. Thus all we have to show is that for arbitrary f and $\varepsilon>0$, $E_{f,\varepsilon}$ is dense. Now let $H\in\mathcal{G}_s$ then

$$H^{-1}\circ\sigma\circ H\in\overline{E_{f,\varepsilon}}\Leftrightarrow\sigma\in H\overline{E_{f,\varepsilon}}H^{-1}\Leftrightarrow\sigma\in\overline{E_{f\circ H,\varepsilon}}.$$

Thus as in the proof of Theorem 1 it suffices to show that for all $\varepsilon>0$ and continuous functions f, $\sigma\in\overline{E_{f,\varepsilon}}$. We formulate this as

3.1. Lemma. *Given $\delta>0$ there exists $G\in\mathcal{G}_s$ such that*

(1) $d(\sigma,G^{-1}\circ\sigma\circ G)<\delta$,

(2) $G^{-1}\circ\sigma\circ G\in E_{f,\varepsilon}$.

Observe first that if $G \in \mathcal{G}_s$ corresponds to the continuous map $z \to g_z$ of Z into $\mathcal{G}$ then

$$\frac{1}{n+1}\sum_{k=0}^{n} f(G^{-1}\circ\sigma^k\circ G(z_0, y_0)) = \frac{1}{n+1}\sum_{k=0}^{n} f(\sigma^k z_0, g_{\sigma^k z_0}^{-1} g_{z_0}(y_0)).$$

The strict ergodicity of (Z,σ) implies that for all y_0 and z_0 this expression tends to

$$\int f(z, g_z^{-1}(g_{z_0}(y_0))d\mu(z),$$

where μ is the unique invariant measure on Z. Since the family of functions $\{F_{(z_0, y_0)}\}_{(z_0,y_0)\in Z\times Y}$ where

$$F_{z_0,y_0}: z \to f(z, g_z^{-1}(g_{z_0}(y_0)),$$

is compact the above convergence is uniform in $x_0 = (z_0, y_0) \in X$.

Thus in order to prove the existence of G as in the lemma, it suffices to show that there exists a continuous map $z \to g_z$ of Z into $\mathcal{G}$ such that (1) holds and in addition for some constant c,

$$\left| \int f(z, g_z^{-1}(y))d\mu(z) - c \right| < \varepsilon, \qquad \forall y \in Y.$$

We assume first that $(Y, \mathcal{G})$ has property (A).

Let $V \neq \emptyset$ be an open subset of Y such that for every v, $v' \in V$

$$\operatorname*{Sup}_z |f(z,v) - f(z,v')| < \varepsilon/5.$$

3.2. Lemma. *There exists a continuous map $t \to h_t$ of I into $\mathcal{G}$ such that for all $y \in Y$*

$$\lambda\{t: h_t^{-1}(y) \in Y\setminus V\} < \frac{\varepsilon}{10\|f\|} = \gamma.$$

Proof. Let $\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_M \in \mathcal{G}$ satisfy the condition

$$\frac{1}{M}\sum 1_{\tilde{h}_i(Y\setminus V)}(z) \leqq \frac{\gamma}{4}.$$

Define $h_t = \tilde{h}_i$ for $(i-1)/M + 1/M^2 \leqq t \leqq i/M - 1/M^2$ $(1 \leqq i \leqq M)$ and extend the map $t \to h_t$ continuously to all of I.

If $y \notin \bigcup_{i=1}^{M} \tilde{h}_i(Y\setminus V)$ then for every

$$t \in \bigcup_{i=1}^{M} [(i-1)/M + 1/M^2, i/M - 1/M^2],$$

$h_t^{-1}(y) \notin Y \backslash V$ and

$$\lambda\{t \colon h_t^{-1}(y) \in Y \backslash V\} \leqq 2M \cdot \frac{1}{M^2} = \frac{2}{M} < \frac{\gamma}{2} < \gamma.$$

If $y \in \bigcap_{j=1}^{l} \tilde{h}_{i_j}(Y \backslash V)$ then

$$\frac{1}{M} \sum_{k=1}^{M} 1_{\tilde{h}_k(Y \backslash V)}(y) = \frac{l}{M} < \frac{\gamma}{2}$$

and

$$\lambda\{t \colon h_t^{-1}(y) \in Y \backslash V\} \leqq \frac{2}{M} + \frac{l}{M} < \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma.$$

This proves Lemma 3.2.

PROOF OF LEMMA 3.1. We can now proceed with the construction of G. Let $\eta > 0$ be such that $|t_1 - t_2| < \eta$ implies $d(h_{t_1}^{-1} h_{t_2}, \mathrm{id}) < \delta$ and let $N \in \mathbf{N}$ with $1/N < \min(\eta/2, \gamma)$.

There exists a measurable subset A of Z with $\mu(A) > 0$, such that A, $\sigma A, \cdots, \sigma^{N^2-1} A$ are pairwise disjoint and such that $1 - \mu(\bigcup_{i=0}^{N^2-1} \sigma^i A) < \gamma$. Let $K \subset A$ be a closed subset homeomorphic to a Cantor set, for which $\mu(A \backslash K) < \gamma / N^2$. Let $\tilde{\theta} \colon K \to I$ be a continuous onto map for which

$$\tilde{\theta}\left(\frac{\mu \mid K}{\mu(K)}\right) = \lambda.$$

Define $\tilde{\theta}$ on $\bigcup_{i=0}^{N^2-1} \sigma^i K$ by

$$\tilde{\theta}(z) = \tilde{\theta}(\sigma^{-i} z) \qquad \text{if } z \in \sigma^i K \; (i = 1, \cdots, N^2 - 1)$$

and extend it to a continuous map $\tilde{\theta} \colon Z \to I$. Finally put

$$\theta(z) = \frac{1}{N} \sum_{i=0}^{N-1} \tilde{\theta}(\sigma^i z) \qquad \text{and} \qquad g_z = h_{\theta(z)}.$$

Thus $G(z, y) = (z, g_z(y)) = (z, h_{\theta(z)}(y))$.

Let $v_0 \in V$; we claim that for every $y \in Y$

$$\left| \int f(z, g_z^{-1}(y)) d\mu(z) - \int f(z, v_0) d\mu(z) \right| < \varepsilon.$$

In fact

$$\int |f(z, g_z^{-1}(y)) - f(z, v_0) d\mu(z)|$$

$$\leqq \frac{\varepsilon}{5} + \int_{\bigcup_{i=0}^{N^2-1} \sigma^i A} |f(z, g_z^{-1}(y)) - f(z, v_0)| d\mu(z)$$

$$\leqq \frac{\varepsilon}{5} + N^2 \frac{\varepsilon}{N^2 \cdot 5} + \int_{\bigcup_{i=0}^{N^2-1} \sigma^i K} |f(z, g_z^{-1}(y)) - f(z, v_0)|\, d\mu(z)$$

$$\leqq \frac{2\varepsilon}{5} + N\mu(K) 2\|f\| + \int_{\bigcup_{i=0}^{N^2-N-1} \sigma^i K} |f(z, g_z^{-1}(y)) - f(z, v_0)|\, d\mu(z)$$

$$= \frac{2\varepsilon}{5} + N\mu(K) 2\|f\| + \sum_{i=0}^{N^2-N-1} \int_K |f(\sigma^i z, h_{\theta(\sigma^i z)}^{-1}(y)) - f(\sigma^i z, v_0)|\, d\mu(z).$$

Now for $z \in K$ and $0 \leqq i \leqq N^2 - N - 1$, $\theta(\sigma^i z) = \tilde{\theta}(z)$, also for $z \in K$ for which $h_{\tilde{\theta}(z)}^{-1}(y) \notin Y \setminus V$ the integrand is $\leqq \varepsilon/5$, and we conclude that the last line above is

$$\leqq \frac{2\varepsilon}{5} + N\mu(K) 2\|f\| + (N^2 - N - 1) 2\|f\| \mu\{z \in K : h_{\tilde{\theta}(z)}^{-1}(y) \in Y \setminus V\} + \varepsilon/5.$$

But

$$\mu\{z \in K : h_{\tilde{\theta}(z)}^{-1}(y) \in Y \setminus V\} = \lambda\{t \in I : h_t^{-1}(y) \in Y \setminus V\} \cdot \mu(K) < \gamma \cdot \mu(K).$$

If we recall that $\mu(K) < 1/N^2$ and that $\gamma = \varepsilon/10\|f\|$, we finally have for every $y \in Y$

$$\left| \int f(z, g_z^{-1}(y)) d\mu(z) - \int f(z, v_0) d\mu(z) \right| < \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.$$

This shows that G satisfies (2). In order to show that it satisfies (1) we observe that $|\theta(\sigma z) - \theta(z)| \leqq 2/N < \eta$. It follows that $d(g_{\sigma z}^{-1} g_z, \mathrm{id}) = d(h_{\theta(\sigma z)}^{-1} h_{\theta(z)}, \mathrm{id}) < \delta$ and hence $d(G^{-1} \circ \sigma \circ G, \sigma) < \delta$. This completes the proof of Lemma 3.1 and hence also of Theorem 2, under the assumption that $(Y, \mathcal{G})$ satisfies the condition (A).

We now assume that $\mathcal{G}$ is compact, acts transitively on Y and that there exists a continuous map $t \to h_t$ which maps I onto $\mathcal{G}$. We can assume that this map sends λ onto Haar measure on $\mathcal{G}$, say ν [10].

Let $\eta > 0$ be such that $|t_1 - t_2| < \eta$ implies $d(h_{t_1}^{-1} h_{t_2}, \mathrm{id}) < \delta$, and let $N \in \mathbf{N}$ with $1/N < \min(\eta/2, \gamma)$.

There exists a measurable subset A of Z with $\mu(A) > 0$, such that A,

$\sigma A, \cdots, \sigma^{N^2-1}A$ are pairwise disjoint, and such that $1-\mu(\bigcup_{i=0}^{N^2-1}\sigma^i A)<\gamma$.

Since $f(z,y)$ is uniformly continuous there exists $\alpha>0$ such that $|z_1-z_2|<\alpha$ implies $|f(z_1,y)-f(z_2,y)|<\varepsilon/5$, for all $y \in Y$. Let $K_1, K_2, \cdots, K_M$ be disjoint subsets of A each homeomorphic to a Cantor set such that $\mu(K_j)>0$, $1 \leqq j \leqq M$, and

$$\mu\left(A \setminus \bigcup_{j=1}^{M} K_j\right) < \frac{\varepsilon}{2\|f\|5N^2},$$

and for every $1 \leqq j \leqq M$ and $0 \leqq i \leqq N^2-1$, $\operatorname{diam}(\sigma^i K_j)<\alpha$.

For every j let $\tilde{\theta}: K_j \to I$ be a continuous onto map for which

$$\tilde{\theta}\left(\frac{\mu \mid K_j}{\mu(K_j)}\right)=\lambda.$$

Define $\tilde{\theta}$ on $\bigcup_{j=1}^{M} \bigcup_{i=0}^{N^2-1}\sigma^i K_j$ by

$$\tilde{\theta}(z)=\tilde{\theta}(\sigma^{-i}z) \qquad \text{if } z \in \sigma^i K_j,$$

and extend it to a continuous map $\tilde{\theta}: Z \to I$. Put

$$\theta(z)=\frac{1}{N}\sum_{i=0}^{N-1}\tilde{\theta}(\sigma^i z) \qquad \text{and} \qquad g_z=h_{\theta(z)}.$$

For every j, $1 \leqq j \leqq M$, choose an arbitrary point $z_j \in K_j$. Let

$$c=\sum_{j=1}^{M}\sum_{i=0}^{N^2-1}\mu(K_j)\int_{\mathscr{G}} f(\sigma^i z_j, g^{-1}(y))d\nu(g),$$

where ν is Haar measure on $\mathscr{G}$. Since $\mathscr{G}$ acts transitively, c is a constant which does not depend on y. We claim that for every $y \in Y$

$$\left|\int f(z, g_z^{-1}(y))d\mu(z)-c\right|<\varepsilon.$$

In fact, as above

$$\left|\int f(z,g_z^{-1}(y))d\mu(z)-c\right| \leqq \frac{3\varepsilon}{5}+\left|\int_{\bigcup_{j=1}^{M}\bigcup_{i=0}^{N^2-N-1}\sigma^i K_j} f(z,g_z^{-1}(y))d\mu(z)-c\right|$$

$$\leqq \frac{3\varepsilon}{5}+\sum_{i=0}^{N^2-N-1}\sum_{j=1}^{M}\left|\int_{K_j} f(\sigma^i z, h^{-1}_{\theta(\sigma^i z)}(y))d\mu(z)-\mu(K_j)\int_{\mathscr{G}} f(\sigma^i z_j, g^{-1}(y))d\nu(g)\right|.$$

But

$$\int_{\mathscr{G}} f(\sigma^i z_j, g^{-1}(y))d\nu(g)=\int_I f(\sigma^i z_j, h_t^{-1}(y)d\lambda(t)$$

$$=\frac{1}{\mu(K_j)}\int_{K_j} f(\sigma^i z_j, h^{-1}_{\theta(z)}(y)d\mu(z),$$

and therefore we have

$$\left|\int f(z, g_z^{-1}(y))d\mu(z)-c\right| \leqq$$

$$\frac{3\varepsilon}{5}+\sum_{i=0}^{N^2-N-1}\sum_{j=1}^{M}\int_{K_j}|f(\sigma^i z, h^{-1}_{\theta(\sigma^i z)}(y))-f(\sigma^i z_j, h^{-1}_{\theta(\sigma^i z)}(y))|d\mu(y).$$

Since $\operatorname{diam}(K_j)<\alpha$ this is

$$\leqq\frac{3\varepsilon}{5}+\sum_{i=0}^{N^2-N-1}\frac{\varepsilon}{5}\mu(\cup K_j)\leqq\frac{4\varepsilon}{5}<\varepsilon.$$

This completes the proof of Theorem 2.

4. Relative proximality and weak mixing

A PROOF OF THEOREM 3. Fix $z_0\in Z$ and let U, V be nonempty open subsets of Y: let $\varepsilon>0$ and put

$$E_{U,V,\varepsilon}=\{T\in\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}\colon \exists k \ \operatorname{diam}(T^k((\{z_0\}\times U)\cup(\{z_0\}\times V)))<\varepsilon\}.$$

Clearly $E_{U,V,\varepsilon}$ is an open subset of $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$. By our assumption there exists a finite covering $\{U_i\times V_i\}_{i=1}^N$ of $Y\times Y$ s.t. for every i and every $\varepsilon>0$ there exists $h\in\mathcal{G}$ with $\operatorname{diam}(h(U_i\cup V_i))<\varepsilon$. Now every element T of $\mathcal{R}=\bigcap_{i=1}^N\bigcap_{n=1}^\infty E_{U_i,V_i,1/n}$ generates a flow (X,T) which is a skew extension of (Z,σ) with the property that any pair of points in the fiber over z_0 is proximal. Since (Z,σ) is minimal this implies that the extension $(X,T)\to(Z,\sigma)$ is proximal. Thus we have to show that given a pair U, V of non-empty open sets with the property that for every $\theta>0$ there exists $h\in\mathcal{G}$ with $\operatorname{diam} h(U\cup V)<\theta$, and $\varepsilon>0$, $E_{U,V,\varepsilon}$ is dense in $\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$. Let $H\in\mathcal{G}_s$, then it suffices to show that $H^{-1}\circ\sigma\circ H\in\overline{E_{U,V,\varepsilon}}$ or that $\sigma\in\overline{HE_{U,V,\varepsilon}H^{-1}}$. Now

$$HE_{U,V,\varepsilon}H^{-1}=\{T\in\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}\colon \exists k, \operatorname{diam}(H^{-1}\circ T^k((\{z_0\}\times h_{z_0}U)\cup(\{z_0\}\times h_{z_0}V))) <\varepsilon\}$$

$$\supset\{T\in\overline{\mathcal{S}_{\mathcal{G}}(\sigma)}\colon \exists k, \operatorname{diam}(T^k(\{z_0\}\times h_{z_0}(U\cup V)))<\delta\}$$

$$=E_{h_{z_0}U,h_{z_0}V,\delta},$$

where $z \to h_z$ is the continuous map of Z into $\mathcal{G}$ which defines H, and $\delta > 0$ is such that $d(y_1, y_2) < \delta$ implies $d(h_z^{-1}y_1, h_z^{-1}y_2) < \varepsilon$. It is therefore enough to show that $\sigma \in \overline{E_{U,V,\varepsilon}}$.

The following lemma will prove this.

4.1. LEMMA. *For every $\delta > 0$ there exists $G \in \mathcal{G}_s$ such that*
(1) $d(\sigma, G^{-1} \circ \sigma \circ G) < \delta$,
(2) $G^{-1} \circ \sigma \circ G \in E_{U,V,\varepsilon}$.

PROOF. Let $h_{\frac{1}{2}} \in \mathcal{G}$ with $\operatorname{diam}(h_{\frac{1}{2}}(U \cup V)) < \varepsilon$ and put $h_0 = h_1 = \mathrm{id}$. Let $t \to h_t$ be a continuous extension to a map of I into $\mathcal{G}$. Let $\eta > 0$ be such that $|t_1 - t_2| < \eta$ implies $d(h_{t_1}^{-1}h_{t_2}, \mathrm{id}) < \delta$ and choose $n \in \mathrm{N}$ such that $2/n < \delta$. Let A be a neighborhood of z_0 such that $A, \sigma A, \cdots, \sigma^{n-1}A$ are pairwise disjoint. Let $K \subset A$ be homeomorphic to a Cantor set, and let $\tilde{\theta}\colon K \to I$ be a continuous onto map with $\tilde{\theta}(z_0) = \frac{1}{2}$. Define $\tilde{\theta}$ on $\bigcup_{i=0}^{n-1} \sigma^i K$ by $\tilde{\theta}(z) = \tilde{\theta}(\sigma^{-i}z)$ if $z \in \sigma^i K$; extend to all of Z and put

$$\theta(z) = \frac{1}{n}\sum_{i=0}^{n-1} \tilde{\theta}(\sigma^i z).$$

Put $g_z = h_{\theta(z)}$ and $G(z, y) = (z, g_z(y))$. It is easy to check that $d(G^{-1} \circ \sigma \circ G, \sigma) < \delta$. Now we can choose k such that $\theta(\sigma^k z_0)$ is so close to 1 that $g_{\sigma^k z_0}^{-1}$ is close to the identity map. Since $g_{z_0} = h_{\theta(z_0)} = h_{\frac{1}{2}}$, $G^{-1} \circ \sigma \circ G \in E_{U,V,\varepsilon}$. This completes the proofs of Lemma 4.1 and Theorem 3.

A PROOF OF THEOREM 4. Let

$$L = \{((z, y_1), (z, y_2)) \colon z \in Z, y_1, y_2 \in Y\} \subset X \times X.$$

Let U_1, U_2, V_1, V_2 be four non-empty open sets in X such that $(U_i \times V_i) \cap L \neq \varnothing$ $(i = 1, 2)$. Put

$$E_{U_1, U_2, V_1, V_2} = \{T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)} \colon \exists k, T^k(U_1 \times V_1) \cap (U_2 \times V_2) \cap L \neq \varnothing\}.$$

E_{U_1, U_2, V_1, V_2} is open and if $\{(U_i \times V_i) \cap L\}$ is a basis for open sets in L, then $\mathcal{R} = \bigcap_{i,j} E_{U_i, U_j, V_i, V_j}$ consists precisely of those elements $T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$ for which the extension $(X, T) \to (Z, \sigma)$ is topologically weak mixing. Since for every $G \in \mathcal{G}_s$, $G\bar{E}_{U_1, U_2, V_1, V_2}G^{-1} = \bar{E}_{GU_1, GU_2, GV_1, GV_2}$, all we have to show is that $\sigma \in \bar{E}_{U_1, U_2, V_1, V_2}$. The proof therefore will be completed with the proof of the following lemma.

4.2. LEMMA. *Given $\varepsilon > 0$ there exists $G \in \mathcal{G}_s$ such that*
(1) $d(\sigma, G^{-1} \circ \sigma \circ G) < \varepsilon$,

(2) $G^{-1}\circ\sigma\circ G\in E_{U_1,U_2,V_1,V_2}$.

PROOF. Let U_1', V_1', U_2', V_2' be open sets in Y and $z_1, z_2\in Z$ points such that $\{z_i\}\times U_i'\subset U_i$, $\{z_i\}\times V_i'\subset V_i$ $(i=1,2)$. Since $(Y,\mathcal{G})$ is topologically weak mixing there exists $h\in\mathcal{G}$ such that $h(U_1'\times V_1')\cap(U_2'\times V_2')\neq\emptyset$. Let $h_{\frac{1}{2}}=h$, $h_0=h_1=$ id and let $t\to h_t$ be a continuous extension from I to $\mathcal{G}$. Let $\eta>0$ be such that $|t_1-t_2|<\eta$ implies $d(h_{t_1}^{-1}h_{t_2},\mathrm{id})<\varepsilon$ and let $n\in\mathbf{N}$ with $2/n<\eta$. Let W be a neighbourhood of z_1 for which W, $\sigma W,\cdots,\sigma^{n-1}W$, $\{\sigma^i z_2\}_{i=0}^{n-1}$ are pairwise disjoint. Let $K\subset W$ be a Cantor set containing z_1. Let $\tilde\theta\colon K\to I$ be a continuous onto map with $\tilde\theta(z_1)=\frac{1}{2}$. Extend $\tilde\theta$ to $\bigcup_{i=0}^{n-1}\sigma^i K$ as usual and let $\tilde\theta(z_2)=\tilde\theta(\sigma z_2)=\cdots=\theta(\sigma^{n-1}z_2)=0$. Now extend $\tilde\theta$ to a continuous map $\tilde\theta\colon Z\to I$. Define

$$\theta(z)=\frac{1}{n}\sum_{i=0}^{n-1}\tilde\theta(\sigma^i z),$$

$g_z=h_{\theta(z)}$ and $G(z,y)=(z,g_z(y))$. It is easy to check that G satisfies (1) and (2). The proofs of both Lemma 4.2 and Theorem 4 are completed.

5. Topological weak mixing

A PROOF OF THEOREM 5. Let W_i, $i=1,2,3,4$ be four non-empty open subsets of X and let

$$E_{W_1,W_2,W_3,W_4}=\{T\in\overline{\mathcal{O}(\sigma)}\colon \exists k,\ T^kW_1\cap W_3\neq\emptyset,\ T^kW_2\cap W_4\neq\emptyset\}.$$

Again one easily checks that it suffices to show that $R_\alpha\in\overline{E_{W_1,W_2,W_3,W_4}}$. Thus the proof will be completed with the proof of the following lemma.

5.1. LEMMA. *Given $\varepsilon>0$ there exists $G\in\mathcal{H}(X)$, such that*

(1) $d(R_\alpha, G^{-1}\circ R_\alpha\circ G)<\varepsilon$,
(2) $G^{-1}\circ R_\alpha\circ G\in E_{W_1,W_2,W_3,W_4}$.

PROOF. By the ergodicity of $(Y,\mathcal{G})$ there are $h_1,h_2\in\mathcal{G}$ such that

$$h_1\pi W_1\cap\pi W_3\neq\emptyset\qquad\text{and}\qquad h_2\pi W_2\cap\pi W_4\neq\emptyset,$$

where π is the projection of X onto Y. Let $(t_i,y_i)\in W_i$, $i=1,2,3,4$, with $h_1(y_1)=y_3$, $h_2(y_2)=y_4$ and such that $i\neq j$ implies $t_i\neq t_j$ and $y_i\neq y_j$. This can be done since Y does not reduce to a single point, and, being connected, each open set of Y is infinite. Let $t\to g_t$ be a continuous map of I into $\mathcal{G}$ with $g_{\frac{1}{4}}=h_1$, $g_{\frac{3}{4}}=h_2$, $g_1=g_{\frac{1}{2}}=g_0=$ id. Let V_i, $i=1,2,3,4$, be pairwise disjoint neighbourhoods of y_i and let p_i be continuous functions on Y such that $0 \leqq p_i \leqq 1$, $p_i(y_i) = 1$ and $\operatorname{Supp}(p_i) \subset V_i$.

Let $\eta > 0$ be such that $d(y, \bar{y}) < \eta$ implies $\max_{1 \leq i \leq 4} |p_i(y) - p_i(\bar{y})| < \varepsilon/4$ and let $\delta > 0$ be such that $|t_1 - t_2| < \delta$ implies $d(g_{t_1}^{-1} g_{t_2}, \mathrm{id}) < \min(\eta, \varepsilon)$. Now choose an integer q for which $|\alpha - p/q| < 1/q^2 < \delta$, for some integer p. Let c_i, $i = 1, 2, 3, 4$, be real numbers between 0 and 1 such that

$$\left.\begin{aligned} t_1 + c_1 &= 1/4q \\ t_2 + c_2 &= 3/4q \\ t_3 + c_3 &= 1/2q \\ t_4 + c_4 &= 1/q \end{aligned}\right\} \pmod 1$$

and define $f(y) = \sum_{i=1}^4 c_i p_i(y)$. We now let

$$G(t, y) = (t + f(y), g_{q(t+f(y))}(y)).$$

One can check that

$$G^{-1} \circ R_\alpha \circ G(t, y) = (t + \alpha + f(y) - f(\bar{y}), \bar{y})$$

where

$$\bar{y} = g_{q(t+\alpha+f(y))}^{-1} g_{q(t+f(y))}(y).$$

Applying G to the points (t_i, y_i), $i = 1, 2, 3, 4$ we have

$$G(t_1, y_1) = (1/4q, y_3), \qquad G(t_3, y_3) = (1/2q, y_3),$$

$$G(t_2, y_2) = (3/4q, y_4), \qquad G(t_4, y_4) = (1/q, y_4).$$

There exists therefore a k such that $R_\alpha^k(1/4q, y_3)$ is close to $(1/2q, y_3)$ and $R_\alpha^k(3/4q, y_4)$ is close to $(1/q, y_4)$. Thus

$$R_\alpha^k G W_1 \cap G W_3 \neq \varnothing \qquad \text{and} \qquad R_\alpha^k G W_2 \cap G W_4 \neq \varnothing$$

and (2) is satisfied. To check (1) we observe that

$$\| q(t + \alpha + f(y)) - q(t + f(y)) \| = \| q\alpha \| < \delta$$

where $\|\cdot\|$ denotes distance from the nearest integer. This implies that $d(y, \bar{y}) < \eta$ where $\bar{y} = g_{q(t+\alpha+f(y))}^{-1} g_{q(t+f(y))}(y)$, and

$$|f(y)-f(\bar{y})| \leqq \sum_{i=1}^{4} c_i |p_i(y)-p_i(\bar{y})| < \frac{4\varepsilon}{4} = \varepsilon.$$

Therefore $d(G^{-1} \circ R_\alpha \circ G, R_\alpha) < \varepsilon$ and (1) is satisfied by G as well. This completes the proof.

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